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The effect of rotation on the onset of convection in a horizontal anisotropic porous layer

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Abstract

The effect of rotation and anisotropy on the onset of convection in a horizontal porous layer is investigated using a linear theory and a weak nonlinear theory. The linear theory is based on the usual normal mode technique and the nonlinear theory on the truncated Fourier series analysis. Darcy model extended to include time derivative and Coriolis terms with anisotropic permeability is used to describe the flow through porous media. A modified energy equation including the thermal anisotropy is used. The effect of rotation, mechanical and thermal anisotropy parameters and the Prandtl number on the stationary and overstable convection is discussed. It is found that the effect of mechanical anisotropy is to allow the onset of oscillatory convection instead of stationary. It is also found that the existence of overstable motions in case of rotating porous medium is not restricted to a particular range of Prandtl number as compared to the pure viscous fluid case. The steady finite amplitude analysis is performed using truncated Fourier series to find the Nusselt number. The effect of various parameters on heat transfer is investigated.

Keywords: Rotation; Thermal convection; Anisotropy; Porous medium

1. Introduction

The study of the effect of external rotation on thermal convection has attracted significant experimental and theoretical interest. Because of its general occurrence in geophysical and oceanic flows, it is important to understand how the Coriolis force influences the structure and transport properties of thermal convection. Rotating thermal convection also provides a system to study hydrodynamic instabilities, pattern formation and spatio-temporal chaos in nonlinear dynamical systems. The study of thermal convection in rotating porous media is motivated both theoretically and by its practical applications in engineering. Some of the important areas of applications in engineering include the food processing, chemical process, solidification and centrifugal casting of metals and rotating machinery.

The stability of problems of thermal convection in rotating porous media has been investigated by many authors. Friedrich [1] has investigated the stability of a rotating porous layer heated from below using a linear stability analysis and also a numerical nonlinear analysis. Patil and Vaidyanathan [2] have studied this problem including the influence of variable viscosity. Both these studies considered a non-Darcy model, which is probably subject to the limitations as shown by Nield [3]. Jou and Liaw [4] investigated the problem of thermal convection in a rotating porous layer subject to transient heating from below using Darcy model. They have obtained only the stability conditions for the marginal state. Palm and Tyvand [5] have established an interesting analogy between a rotating porous layer and an anisotropic porous layer. Qin and Kaloni [6] have studied the nonlinear stability of the rotating Benard problem in a porous medium by employing the generalized Brinkman model as a suitable prototype for high porosity porous media using energy theory. They derived sufficient conditions for nonlinear stability and also determined critical energy bound by solving variational problem.

Vadasz [7] has performed a linear as well as a weak nonlinear stability analysis of a rotating porous layer heated from below using Darcy model with time derivative term. It was reported that, in contrast to the problem in pure fluids, overstable

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Nomenclature

a	wavenumber, $\sqrt{l^2+m^2}$	
d	height of the porous layer	m
$\mathbf{D}$	thermal diffusivity tensor, $D_x(\mathbf{ii}+\mathbf{jj})+D_z(\mathbf{kk})$	$\mathrm{m^2\,s^{-1}}$
Da	Darcy number, K_z/d^2	
$\mathbf{g}$	gravitational acceleration, $(0,0,-g)$	$\mathrm{m\,s^{-2}}$
H	rate of heat transport per unit area	$\mathrm{m\,s^{-1}\,K}$
$\mathbf{K}$	permeability tensor, $K_x^{-1}(\mathbf{ii}+\mathbf{jj})+K_z^{-1}(\mathbf{kk})$	$\mathrm{m^{-2}}$
l, m	horizontal wavenumbers	
Nu	Nusselt number	
p	pressure	$\mathrm{kg\,m^{-1}s^{-2}}$
Pr	effective Prandtl number, ν/D_z	
$\mathbf{q}$	velocity vector, (u,v,w)	$\mathrm{m\,s^{-1}}$
R	Rayleigh number, $\beta g\Delta T d K_z/\nu D_z$	
Ra	scaled Rayleigh number, R/π^2	
t	time	s
T	temperature	K
ΔT	temperature difference between the walls	K
Ta	Taylor number, $(2\Omega K_z/\varepsilon\nu)^2$	
x, y, z	space coordinates	m

Greek symbols

α	scaled wavenumber, a^2/π^2	
β	thermal expansion coefficient	$\mathrm{K^{-1}}$
γ	scaled Darcy–Prandtl number, χ/π^2	
χ	Darcy–Prandtl number, $\varepsilon Pr/Da$	
ε	porosity	
η	thermal anisotropy parameter, D_x/D_z	
μ	dynamic viscosity	$\mathrm{kg\,m^{-1}\,s^{-1}}$
ν	kinematic viscosity, μ/ρ_0	$\mathrm{m^2\,s^{-1}}$
θ	dimensionless temperature	
ρ	density	$\mathrm{kg\,m^{-3}}$
σ	growth rate	
τ	rescaled time, χt	
$\boldsymbol{\omega}$	vorticity vector, $\nabla\times\mathbf{q}$	$\mathrm{s^{-1}}$
$\boldsymbol{\Omega}$	angular velocity, $(0,0,\Omega)$	$\mathrm{s^{-1}}$
ξ	mechanical anisotropy parameter, K_x/K_z	
ψ	stream function	$\mathrm{m^2\,s^{-1}}$
∇_h^2	$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$	
∇^2	$\nabla_h^2+\frac{\partial^2}{\partial z^2}$	

Subscripts

b	basic state
c	critical
f	fluid
h	horizontal
0	reference
s	solid

Superscripts

$*$	dimensionless quantity
$'$	perturbed quantity
osc	oscillatory state
st	stationary

convection in porous media at marginal stability is not limited to a particular range of the values of Prandtl number. It was also established by Vadasz [7] that in the porous media problem the wavelength of the roll measured in the plane containing the streamlines is not independent of rotation, a result that is quite distinct from the corresponding pure fluids problem. An excellent review of research on thermal convection in a rotating porous media has been given by Vadasz [8].

A nonlinear stability analysis is performed for thermal convection in a rotating fluid saturated porous layer using energy stability theory by Straughan [9]. It is reported that the global nonlinear stability Rayleigh number is exactly same as that for linear instability and for the rotating porous convection problem governed by Darcy equation, subcritical instabilities are not possible. The effect of Coriolis force on centrifugally driven convection in a rotating porous layer is analyzed by Govender [10]. The marginal stability criterion is established as a characteristic centrifugal Rayleigh number in terms of the wavenumber and the Taylor number. More recently Straughan [11] has analyzed nonlinear stability in porous medium with a thermal nonequilibrium model, when the layer rotates with a constant angular velocity about an axis in the same direction as gravity and demonstrated the equivalence of linear instability and nonlinear stability boundaries.

Anisotropy is generally a consequence of preferential orientation or asymmetric geometry of porous matrix or fibers and is in fact encountered in numerous systems in industry and nature. Anisotropy is particularly important in a geological context, since sedimentary rocks generally have a layered structure; the permeability in the vertical direction is often much less than in the horizontal direction. Anisotropy can also be a characteristic of artificial porous materials like pelleting used in chemical engineering process and fiber material used in insulating purpose. Despite the practical importance, in contexts varying from fibrous insulating material to sedimentary rocks, only few studies have been reported on convection in an anisotropic porous medium uniformly heated from below. The review of research on convective flow through anisotropic porous media has been well documented by McKibbin [12,13] and Storesletten [14,15]. Castinel and Combarnous [16] have conducted an experimental and theoretical investigation on the Rayleigh–Benard convection in an anisotropic porous medium. Epherre [17] extended the stability analysis to a porous medium with anisotropy in thermal diffusivity also. A theoretical analysis of nonlinear thermal convection in an anisotropic porous medium was performed by Kvernvold and Tyvand [18]. Nilsen and Storesletten [19] have studied the problem of natural convection in both isotropic and anisotropic porous channels. Tyvand and Storesletten [20] investigated the problem concerning the onset of convection in an anisotropic porous layer in which the principal axes were obliquely oriented to the gravity vector.

The available works on thermal convection in a rotating porous media are all concerned with isotropic media except a recent work by Govender [21]. The work of Govender [21] is concerned with natural convection in an anisotropic porous layer subject to centrifugal body force employing the Darcy model using linear stability theory. It is found that the convection is stabilized when the anisotropy ratio, which is a function of the thermal and mechanical anisotropy parameters, is increased in magnitude. One of the anonymous reviewers has brought to our notice an unpublished work on the effect of mechanical and thermal anisotropy on the linear stability of stationary convection in a rotating porous media by Govender and Vadasz [22].

The objective of the present study is to investigate the combined effect of rotation and anisotropy on the thermal convection in a horizontal porous layer using a linear analysis and nonlinear analysis.

2. Mathematical formulation

Consider a fluid saturated anisotropic porous layer of infinite horizontal extent confined between parallel, stress-free planes at $z=0$ and $z=d$ subject to rotation and maintained at constant temperatures $T_0+\Delta T$ and T_0 respectively. A Cartesian frame of reference is chosen with x- and y-axes at the lower boundary plane and z-axis directed vertically upwards in the gravity field. The axis of rotation is assumed to coincide with the z-axis. The porous medium is assumed to possess horizontal isotropy in both mechanical and thermal properties. The extended Darcy law, which includes the time derivative and the Coriolis term, is employed as a momentum equation and Boussinesq approximation is applied to account for the effects of density variations. With these assumptions the basic governing equations may be written as

$$\nabla\cdot\mathbf{q}=0 \tag{2.1}$$

$$\frac{1}{\varepsilon}\frac{\partial\mathbf{q}}{\partial t}+\frac{2}{\varepsilon}\boldsymbol{\Omega}\times\mathbf{q}+\frac{\mu}{\rho_0}\mathbf{K}\cdot\mathbf{q}=-\frac{1}{\rho_0}\nabla p+\frac{\rho}{\rho_0}\mathbf{g} \tag{2.2}$$

$$\frac{\partial T}{\partial t}+(\mathbf{q}\cdot\nabla)T=\nabla\cdot(\mathbf{D}\cdot\nabla T) \tag{2.3}$$

$$\rho=\rho_0\big[1-\beta(T-T_0)\big] \tag{2.4}$$

The components of the thermal diffusivity tensor $\mathbf{D}$ are written in terms of porosity and appropriate thermal diffusivities of the fluid and solid states as $D_i=\varepsilon(D_i)_f+(1-\varepsilon)(D_i)_s$ with $i=x,z$.

The basic state of the fluid is assumed to be quiescent. The quantities of the basic state are given by, $\mathbf{q}_b=(0,0,0)$, $T_b=T(z)$, $p_b=p(z)$, $\rho_b=\rho(z)$, which satisfy the equations

$$\frac{dp_b}{dz}=\rho_b g,\qquad \frac{d^2T_b}{dz^2}=0,\qquad \rho_b=\rho_0\big[1-\beta(T_b-T_0)\big] \tag{2.5}$$

On the basic state we superpose small perturbations around the basic solutions in the form

$$\mathbf{q}=\mathbf{q}_b+\mathbf{q}',\qquad T=T_b(z)+T'$$
$$p=p_b(z)+p',\qquad \rho=\rho_b(z)+\rho' \tag{2.6}$$

where the primes indicate perturbations. Substituting Eq. (2.6) into Eqs. (2.1)–(2.4) using the basic state equations (2.5) and the transformations

$$(x,y,z)=d(x^*,y^*,z^*),\qquad t=\frac{d^2}{D_z}t^*,\qquad p'=\frac{\mu D_z}{K_z}p^*$$
$$(u',v',w')=\frac{D_z}{d}(u,v,w),\qquad T'=(\Delta T)T^*$$

to render the equations dimensionless we obtain (after dropping the asterisks for simplicity),

$$\nabla\cdot\mathbf{q}=0 \tag{2.7}$$

$$\frac{\partial\mathbf{q}}{\partial\tau}+Ta^{1/2}\mathbf{k}\times\mathbf{q}+\mathbf{q}_a=-\nabla p+RT\mathbf{k} \tag{2.8}$$

$$\chi\frac{\partial T}{\partial\tau}+(\mathbf{q}\cdot\nabla)T-w=\left[\eta\nabla_h^2+\frac{\partial^2}{\partial z^2}\right]T \tag{2.9}$$

where $\mathbf{q}_a=(\frac{u}{\xi},\frac{v}{\xi},w)$, is the anisotropy modified velocity vector and $\tau=\chi t$ is the rescaled time. The boundary conditions in only the z-direction are required for solving Eqs. (2.7)–(2.9) and are given by

$$w=T=0\quad\text{at } z=0 \text{ and } z=1. \tag{2.10}$$

Vadasz [7] in his comprehensive work reported that the typical values of Darcy–Prandtl number χ in traditional porous media applications are quite big, a fact which provides the justification for neglecting the time derivative term in Eq. (2.8). This is then the classical theory of Darcy. Nield and Bejan [23] argue for this scenario, which is certainly true in many geophysical and engineering applications. However, Vadasz [7] argues that in circumstances linked to modern porous media applications the value of χ can become of unit order of magnitude or even smaller, in which case the time derivative should be retained. Straughan [9] and Govender [10] have also supported this argument. Accordingly, following Vadasz [7] line of argument, in the present paper, we keep the time derivative terms in the Darcy equation in order to allow for the possibility of overstable motions and will find how Darcy–Prandtl number χ influences the overstable motions. It is worth mentioning that the Rayleigh number and Taylor number in the present paper are defined in terms of vertical permeability K_z and they differ by a factor of Da^{-1} and Da^{-2}, respectively, from the corresponding definitions used by Rudraiah et al. [24] to study the effect of rotation on double diffusive convection in a sparsely packed porous medium.

We now eliminate the pressure from Eq. (2.8) by applying the curl operator on it which yields an equation for vorticity, defined as $\boldsymbol{\omega}=\nabla\times\mathbf{q}$, in the form

$$\frac{\partial\boldsymbol{\omega}}{\partial\tau}+\frac{1}{\xi}\boldsymbol{\omega}_a-Ta^{1/2}\frac{\partial\mathbf{q}}{\partial z}=R\left[\frac{\partial T}{\partial y}\mathbf{i}-\frac{\partial T}{\partial x}\mathbf{j}\right] \tag{2.11}$$

where $\boldsymbol{\omega}_a=\nabla\times\mathbf{q}_a$, denotes the anisotropy modified vorticity vector.

It is important to note that the vertical component of Eq. (2.11) is independent of temperature. Once again by applying the curl on Eq. (2.11) one can get the equation

$$\frac{\partial}{\partial\tau}\left(\nabla^2\mathbf{q}\right)+Ta^{1/2}\frac{\partial\boldsymbol{\omega}}{\partial z}+\mathbf{Q}+R\left[\frac{\partial^2 T}{\partial x\partial z}\mathbf{i}+\frac{\partial^2 T}{\partial y\partial z}\mathbf{j}-\nabla_h^2 T\mathbf{k}\right]=0 \tag{2.12}$$

with

$$\mathbf{Q}=(Q_1,Q_2,Q_3)$$

where

$$Q_1=\left[\frac{\partial^2}{\partial x^2}+\frac{1}{\xi}\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\right]u+\left(1-\frac{1}{\xi}\right)\frac{\partial^2 v}{\partial x\partial y}$$

$$Q_2=\left[\frac{\partial^2}{\partial y^2}+\frac{1}{\xi}\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial x^2}\right)\right]v+\left(1-\frac{1}{\xi}\right)\frac{\partial^2 u}{\partial x\partial y}$$

and

$$Q_3=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{1}{\xi}\frac{\partial^2}{\partial z^2}\right)w$$

3. Linear stability analysis

In this section, we perform the linear stability analysis, which is very useful in the local nonlinear stability analysis discussed in the next section. For this, we remove the coupling between the linear form of Eq. (2.9) and Eqs. (2.11)–(2.12) by eliminating $\mathbf{q}$ and $\boldsymbol{\omega}$ to provide a single equation for the temperature perturbation in the form

$$\left\{\left[\frac{\partial}{\partial\tau}\left(\frac{\partial}{\partial\tau}+\frac{1}{\xi}\right)\nabla^2+\left(\frac{\partial}{\partial\tau}+\frac{1}{\xi}\right)\left(\nabla_h^2+\frac{1}{\xi}\frac{\partial^2}{\partial z^2}\right)+Ta\frac{\partial^2}{\partial z^2}\right]\times\left(\chi\frac{\partial}{\partial\tau}-\eta\nabla_h^2-\frac{\partial^2}{\partial z^2}\right)-R\left(\frac{\partial}{\partial\tau}+\frac{1}{\xi}\right)\nabla_h^2\right\}T=0 \tag{3.1}$$

Then the boundary conditions in terms of T are given by

$$T=\frac{\partial^2 T}{\partial z^2}=0\quad\text{at } z=0 \text{ and } z=1 \tag{3.2}$$

We assume the normal mode solution in the form

$$T=\theta(z)\exp\left[i(lx+my)+\sigma\tau\right] \tag{3.3}$$

Using (3.3) in Eq. (3.1) one can obtain an ordinary differential equation for $\theta(z)$ as follows

$$\left\{\left[\sigma\left(\sigma+\xi^{-1}\right)\left(D^2-a^2\right)+\left(\sigma+\xi^{-1}\right)\left(\xi^{-1}D^2-a^2\right)+TaD^2\right]\left(D^2-\eta a^2-\chi\sigma\right)-R\left(\sigma+\xi^{-1}\right)a^2\right\}\theta=0 \tag{3.4}$$

where $D\equiv\frac{d}{dz}$. The boundary conditions (3.2) reduce to

$$\theta=D^2\theta=0\quad\text{at } z=0 \text{ and } z=1 \tag{3.5}$$

The solution of Eq. (3.4) satisfying Eq. (3.5) should be a periodic wave of the form $\theta(z)=A_n\sin(n\pi z)$ which minimizes the Rayleigh number when $n=1$, indicating that $\theta(z)=A_1\sin(\pi z)$ is the eigenfunction for marginal stability. Substituting this into Eq. (3.4) one can obtain

$$Ra=\frac{[1+\eta\alpha+\gamma\sigma][(\sigma+\xi^{-1})\{\sigma(1+\alpha)+(\xi^{-1}+\alpha)\}+Ta]}{\alpha(\sigma+\xi^{-1})} \tag{3.6}$$

an expression for the scaled Rayleigh number $Ra=R/\pi^2$, where $\alpha=a^2/\pi^2$ and $\gamma=\chi/\pi^2$ being the rescaled wavenumber and Darcy–Prandtl number respectively. One can observe that when $\xi=\eta=1$, Eq. (3.6) yields the result of Vadasz [7] for the case of isotropic porous layer.

If σ is real, then marginal stability occurs when $\sigma=0$. Then Eq. (3.6) gives the Rayleigh number Ra^{st} for the onset of stationary convection, in the form

$$Ra^{\mathrm{st}}=\frac{1}{\alpha}\left(\xi^{-1}+\alpha\right)(\eta\alpha+1)+\frac{\xi}{\alpha}(\eta\alpha+1)Ta \tag{3.7}$$

The first term in the expression for stationary Rayleigh number represents the value for the onset of convection in the absence of rotation while the second term represents the contribution of rotation. The critical value of stationary Rayleigh number and the corresponding wavenumber is given by

$$Ra_c^{\mathrm{st}}=\left[1+\sqrt{\xi\eta\left(\xi^{-2}+Ta\right)}\right]^2\quad\text{and}\quad \alpha_c^{\mathrm{st}}=\left[\xi\eta^{-1}\left(\xi^{-2}+Ta\right)\right]^{1/2} \tag{3.8}$$

The problem of stationary convection has been studied by Govender and Vadasz [22] and for details one can refer to the work of Govender and Vadasz [22]. We discuss below in detail the oscillatory convection.

It is well known that the oscillatory motions are possible only if some additional constraints like rotation, salinity gradient and magnetic field are present. For the oscillatory mode σ must be represented as $\sigma=\sigma_r+i\sigma_i$. At the marginal state $\sigma_r=0$ and $\sigma_i\neq 0$. Substituting $\sigma=i\sigma_i$ into Eq. (3.6) and imposing the condition $\sigma_i^2>0$, which is the requirement for σ_i to be real in order to get overstability possible at all, yields two algebraic equations by requiring the imaginary and the real part of Eq. (3.6) to vanish separately. This provides the solution for the characteristic values of the Rayleigh number and the frequency σ_i of the oscillations at the margin of stability in the form

$$Ra^{\mathrm{osc}}=\frac{2+\alpha(1+\xi)}{\alpha\xi^2}\times\left[\gamma\xi(\eta\alpha+1)(2+\alpha+\alpha\xi)+\xi^2(1+\alpha)(\eta\alpha+1)^2+\gamma^2(1+\alpha\xi+\xi^2 Ta)\right]\left[\xi(1+\alpha)(\eta\alpha+1)+\gamma(1+\alpha\xi)\right]^{-1} \tag{3.9}$$

$$\sigma_i^2=\frac{[\xi(\eta\alpha+1)-\gamma]Ta}{\xi(1+\alpha)(\eta\alpha+1)+\gamma(1+\alpha\xi)}-\frac{1}{\xi^2} \tag{3.10}$$

The critical Rayleigh number, wavenumber and corresponding frequency are obtained by minimizing Ra^{osc} in Eq. (3.9) with respect to α which results in a sixth degree equation for α in the form

$$\alpha^6+\Delta_1\alpha^5+\Delta_2\alpha^4+\Delta_3\alpha^3+\Delta_4\alpha^2+\Delta_5\alpha+\Delta_6=0 \tag{3.11}$$

(the expressions for the coefficients are not given for brevity). The solution to Eq. (3.11) is obtained numerically which gives one real positive root α_c^{osc} that minimizes the Rayleigh number Ra^{osc} corresponding to each set of values of γ, Ta, ξ and η within the overstability limit. By substituting the value of α_c^{osc} into Eq. (3.9) one obtains the critical Rayleigh number for oscillatory mode. Similarly, substituting the critical wavenumber α_c^{osc} into Eq. (3.10) we obtain the critical frequency $\sigma_{i,c}^2$.

It has been investigated by Vadasz [7] that there is no straight limitation on the Prandtl number for the overstability to set in at the threshold in the case of thermal convection in a rotating porous layer. It is worth mentioning that in case of pure fluid problem with rotation, the Prandtl number should be less than unity for the overstability to set in at the onset of convection. This limits the inventory of fluids for which convection can set in as overstability in the pure fluid case. It is interesting to note from Eq. (3.10) that there is no straight limitation on the Darcy–Prandtl number for overstability to set in at the onset of convection in an anisotropic porous medium also. However, Eq. (3.10) gives a condition relating the Darcy–Prandtl number and the Taylor number, which permits overstability in the form

$$Ta > \frac{(1+\alpha)(\eta\alpha+1)+\gamma\xi^{-1}(1+\alpha\xi)}{\xi[\xi(\eta\alpha+1)-\gamma]} \quad \text{and} \\ \gamma < \xi(\eta\alpha+1) \tag{3.12}$$

It is worth noticing the particular case when $\gamma \to 0$, which demands some special attention because it provides a lower bound for the overstable characteristic curves. Accordingly substituting $\gamma = 0$ in Eqs. (3.9) and (3.10) we obtain

$$Ra^{osc} = \frac{[2+\alpha(1+\xi)](\eta\alpha+1)}{\alpha\xi} \quad \text{and} \\ \sigma_i^2 = \frac{Ta}{(1+\alpha)} - \frac{1}{\xi^2} \tag{3.13}$$

It is easy to verify that the overstable Rayleigh number corresponding to $\gamma \to 0$, given by Eq. (3.13) takes minimum value when $\alpha = \alpha_c$ where α_c is given by

$$\alpha_c = \frac{1}{\alpha\eta}\left[-\xi\eta + \sqrt{\xi^2\eta^2 + 8\eta}\right] \tag{3.14}$$

It is evident from Eq. (3.13) that the characteristic curves for $\gamma = 0$ are all independent of the Taylor number. Therefore they provide the lower limit for all characteristic curves. The characteristic curves corresponding to different values of γ lie in between the curve for $\gamma = 0$ and the stationary convection curve associated with particular values of the other parameters. When $\xi = \eta = 1$, Eq. (3.13) reduces to

$$Ra^{osc} = \frac{2(1+\alpha)^2}{\alpha} \quad \text{and} \quad \sigma_i^2 = \frac{Ta}{(1+\alpha)} - 1 \tag{3.15}$$

Substituting $\xi = \eta = 1$ in Eqs. (3.9) and (3.10) one can recover the results of isotropic case (Vadasz [7])

$$Ra^{osc} = \frac{2}{\alpha}\left[(1+\alpha)(1+\alpha+\gamma) + \frac{\gamma^2 Ta}{1+\alpha+\gamma}\right] \tag{3.16}$$

$$\sigma_i^2 = \frac{(1+\alpha-\gamma)Ta}{(1+\alpha)(1+\alpha+\gamma)} - 1 \tag{3.17}$$

It has been established by Straughan [9] that the global nonlinear stability Rayleigh number is exactly the same as that for linear instability given by Vadasz [7] and for the rotating porous convection problem governed by the Darcy equation, subcritical instabilities are not possible. Therefore it is believed that in case of an anisotropic rotating porous convection also the subcritical instabilities are not possible and the linear stability theory completely captures the physics, describing the onset of convection. In the next section we perform a nonlinear stability analysis and express heat transfer by conduction and convection and observe the effect of rotation through Ta on it.

4. Finite amplitude steady convection with limited representation

In this section we consider the nonlinear analysis using a truncated representation of Fourier series with only two terms. Although the linear stability analysis is sufficient for obtaining the stability condition of the motionless solution and the corresponding eigenfunctions describing qualitatively the convective flow, it cannot provide information about the values of the convection amplitudes, nor regarding the rate of heat transfer. To obtain this additional information, we perform the nonlinear analysis, which is useful to understand the physical mechanism with minimum amount of mathematical analysis and is a step forward towards understanding full nonlinear problem.

For simplicity of analysis, we confine ourselves to the two-dimensional rolls, so that all the physical quantities are independent of y. We introduce stream function such that $u = \partial\psi/\partial z$, $w = -\partial\psi/\partial x$ into Eqs. (2.8)–(2.9) and setting $\frac{\partial}{\partial t} = 0$ (for the steady state) to obtain

$$\left[\left(\frac{\partial^2}{\partial x^2} + \frac{1}{\xi}\frac{\partial^2}{\partial z^2}\right) + \xi\, Ta \frac{\partial^2}{\partial z^2}\right]\psi + R\frac{\partial T}{\partial x} = 0 \tag{4.1}$$

$$\left(\eta\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial z^2}\right)T + \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x} = 0 \tag{4.2}$$

A minimal double Fourier series which describes the finite amplitude steady convection is given by

$$\psi = A\sin(ax)\sin(\pi z) \tag{4.3}$$

$$T = B\cos(ax)\sin(\pi z) + C\sin(2\pi z) \tag{4.4}$$

where the amplitudes A, B and C are constants and are to be determined from the dynamics of the system. Substituting Eqs. (4.3)–(4.4) into Eqs. (4.1)–(4.2) and equating the coefficients of like terms we obtain the following nonlinear system of equations

$$\left[\left(a^2 + \frac{\pi^2}{\xi}\right) + \pi^2\xi\, Ta\right]A + aRB = 0 \tag{4.5}$$

$$aA + (\eta a^2 + \pi^2)B + \pi aAC = 0 \tag{4.6}$$

$$8\pi^2 C - \pi aAB = 0 \tag{4.7}$$

The steady state solutions are useful because they predict that a finite amplitude solution to the system is possible for subcritical values of the Rayleigh number and that the minimum values of R for which a steady state solution is possible lies below the critical values for instability to either a marginal state or an overstable infinitesimal perturbation. Elimination of all amplitudes, except for A, yields

$$A\left\{\left[\left(a^2+\frac{\pi^2}{\xi}\right)+\pi^2\xi\,Ta\right] - a^2R\left[\left(\eta a^2+\pi^2\right)+a^2\left(\frac{A^2}{8}\right)\right]^{-1}\right\}=0 \tag{4.8}$$

The solution $A=0$ corresponds to pure conduction, which we know to be a possible solution though it is unstable when R is sufficiently large. The remaining solutions are given by

$$\frac{A^2}{8}=\frac{Ra}{\alpha+1/\xi+\xi\,Ta}-\frac{\eta\alpha+1}{\alpha} \tag{4.9}$$

where Ra and α stand for the rescaled Rayleigh number and the wavenumber respectively. Once we know the amplitude we can find the heat transfer.

In the study of convection in porous medium, the quantification of heat transports is important. This is because onset of convection, as Rayleigh number is increased, is more readily detected by its effect on the heat transport. If H is the rate of heat transport per unit area, then

$$H=-D_z\left\langle\frac{\partial T_{\text{total}}}{\partial z}\right\rangle_{z=0} \tag{4.10}$$

where the angular bracket denotes horizontal average and

$$T_{\text{total}}=T_0-\Delta T\frac{z}{d}+T(x,z,t) \tag{4.11}$$

Substituting Eq. (4.4) into Eq. (4.11) and the resulting equation into Eq. (4.10), performing the average, we get

$$H=\frac{D_z\Delta T}{d}(1-2\pi C) \tag{4.12}$$

The Nusselt number Nu is defined by

$$Nu=\frac{H}{D_z\Delta T/d}=1-2\pi C \tag{4.13}$$

Writing C in terms of A, substituting into Eq. (4.13) and using Eq. (3.7), we obtain a simple expression for the Nusselt number in the form

$$Nu=1+2\left(1-\frac{Ra^{\text{st}}}{Ra}\right) \tag{4.14}$$

The second term on the right-hand side of Eq. (4.14) represents the convective contribution to heat transport.

5. Results and discussion

The neutral stability curves in the $Ra-\alpha$ plane for various parameter values are shown in Figs. 1–4. From these figures it is clear that the neutral curves are connected in a topological sense. This connectedness allows the linear stability criteria to be expressed in terms of the critical Rayleigh number, Ra_c, below which the system is stable and unstable above. The points where the overstable solutions branch off from the stationary convection can be easily identified from these figures. Also we observe that for smaller values of the wavenumber each curve is a margin of the oscillatory instability and at some fixed α

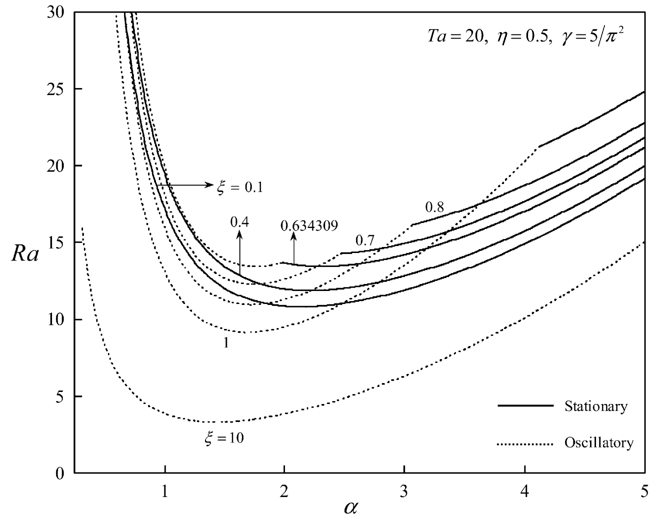

Fig. 1. Neutral stability curves for different values of mechanical anisotropy parameter ξ.

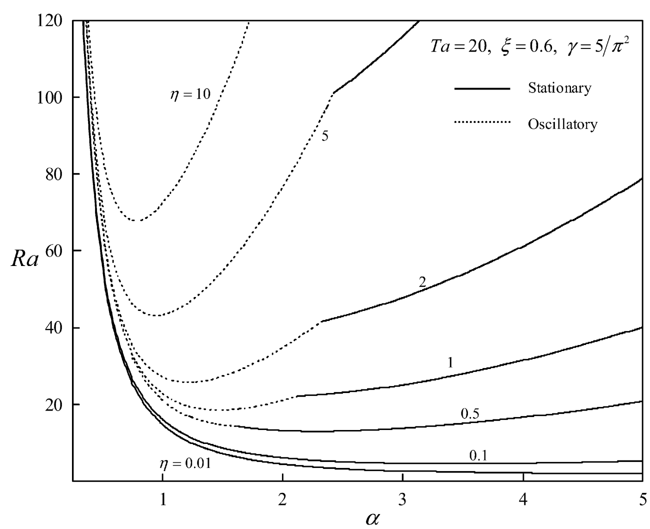

Fig. 2. Neutral stability curves for different values of thermal anisotropy parameter η.

depending on the other parameters the overstability disappears and the curve forms the margin of stationary convection.

The effect of mechanical anisotropy parameter ξ for the fixed values of thermal anisotropy parameter $\eta=0.5$, Taylor number $Ta=20$ and the scaled Darcy–Prandtl number $\gamma=5/\pi^2$, on the marginal stability curves is depicted in Fig. 1. It can be observed that an increase in ξ decreases the minimum of the Rayleigh number for oscillatory state. That is the effect of increasing the ratio of permeability ξ is to advance the onset of oscillatory convection. Further an important question is whether, under the critical conditions for the onset of instability, the instability manifests itself as stationary convection or as oscillatory convection. It is interesting to note that there is a critical value $\xi=\xi^*$ (e.g., for a fixed values of $Ta=20$, $\eta=0.5$, $\gamma=5/\pi^2$, $\xi^*=0.634309$ see Fig. 1) such that for $\xi<\xi^*$ the instability manifested as stationary convection and for $\xi\geqslant\xi^*$, the onset of instability manifests as oscillatory convection. Thus

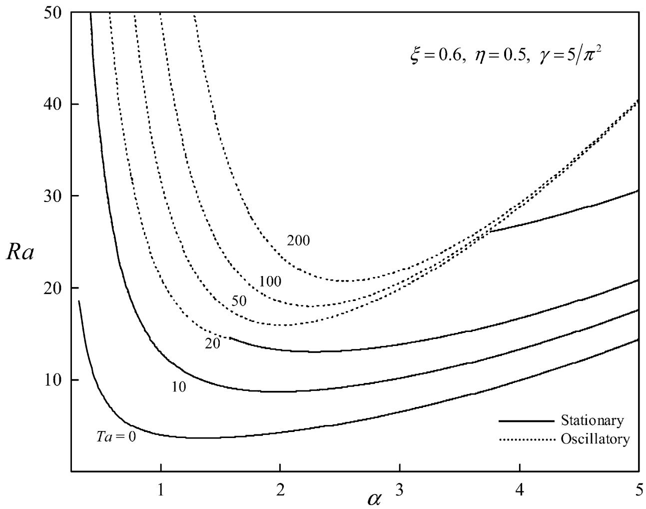

Fig. 3. Neutral stability curves for different values of Taylor number *Ta*.

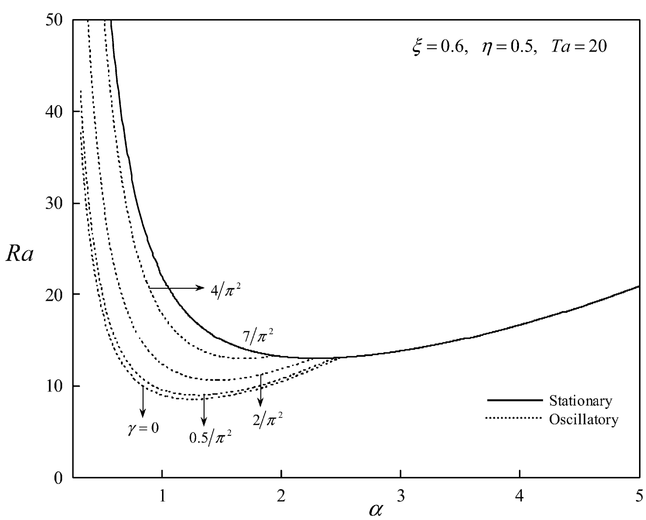

Fig. 4. Neutral stability curves for different values of scaled Darcy–Prandtl number γ.

the effect of mechanical anisotropy is to allow the onset of oscillatory convection instead of stationary convection.

Fig. 2 indicates the effect of thermal anisotropy parameter η on the neutral curves for the fixed values of the mechanical anisotropy parameter $\xi = 0.6$, Taylor number $Ta = 20$ and the scaled Darcy–Prandtl number $\gamma = 5/\pi^2$. It is observed that critical value of Rayleigh number increases with η, indicating that the effect of thermal anisotropy parameter η is to inhibit the onset of convection. We also find from Fig. 2 that the onset of instability manifests as stationary convection for very small values of the thermal anisotropy parameter η. However as η increases, instability sets in as oscillatory mode.

Fig. 3 depicts the effect of rotation on the neutral curves for fixed values $\xi = 0.6$, $\eta = 0.5$ and $\gamma = 5/\pi^2$. We find that the effect of increasing *Ta* is to increase the critical value of the Rayleigh number and the corresponding wavenumber implying that the rotation has a stabilizing effect on the thermal convection in porous medium. This can be explained as follows: rotation acts so as to suppress vertical motion, and hence thermal

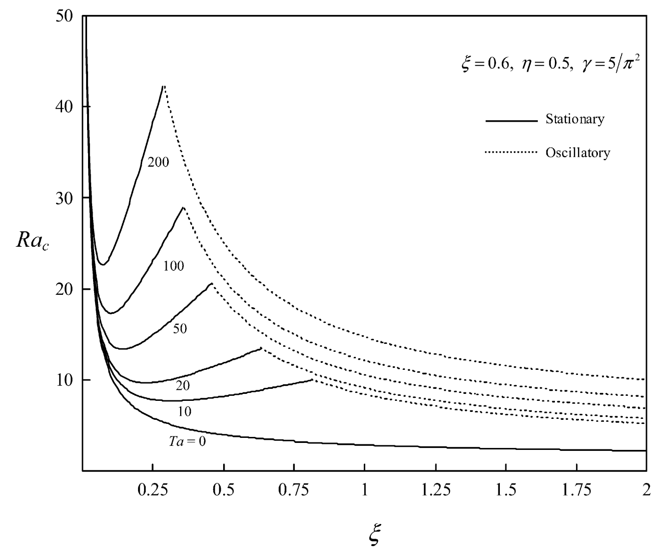

Fig. 5. Variation of scaled critical Rayleigh number with mechanical anisotropy parameter ξ for different values of Taylor number *Ta*.

convection, by restricting the motion to the horizontal plane. Further this figure also indicates that for small *Ta* the instability manifests as stationary convection while as *Ta* is increased, the instability sets in as oscillatory convection. The neutral stability curves corresponding to $Ta = 20$, $\xi = 0.6$, $\eta = 0.5$ and for different values of scaled Darcy–Prandtl number γ are presented in Fig. 4. The points where the overstable solutions branch off from the stationary convection curves can be identified clearly. From this figure it is evident that the characteristic curve for $\gamma = 0$ provide the lower limit for all other curves. The neutral curves corresponding to different values of γ lies between the curve for $\gamma = 0$ and the stationary convection curve when the other parameters are fixed.

The behavior of the stationary and oscillatory critical Rayleigh number as a function of the mechanical anisotropy parameter for different values of Taylor number is shown in Fig. 5. In the absence of rotation i.e., when $Ta = 0$, an increased mechanical anisotropy parameter reduces the stationary critical Rayleigh number. This is the classical result of Epherre [17]. However, in the presence of rotation, it is interesting to note that, the stationary critical Rayleigh number decreases to its minimum value with increasing ξ up to a certain value $\xi = \xi_c$ and as ξ is increased further beyond ξ_c, the critical Rayleigh number for stationary mode increases. Further the effect of anisotropy parameter on the stationary critical Rayleigh number is significant for large Taylor number. The critical Rayleigh number for oscillatory mode however decreases with increasing ξ. This figure also indicates the stabilizing effect of the rotation.

The variation of the critical Rayleigh Ra_c with Taylor number *Ta* for different values of ξ, η and γ is shown in Figs. 6(a)–(c). We observe from these figures that the critical Rayleigh number increases with increase in *Ta* indicating that the effect of rotation is to inhibit the onset of thermal convection and it is in agreement with the corresponding problem of isotropic porous layer (Vadasz [7]) and pure fluid layer (Chan-

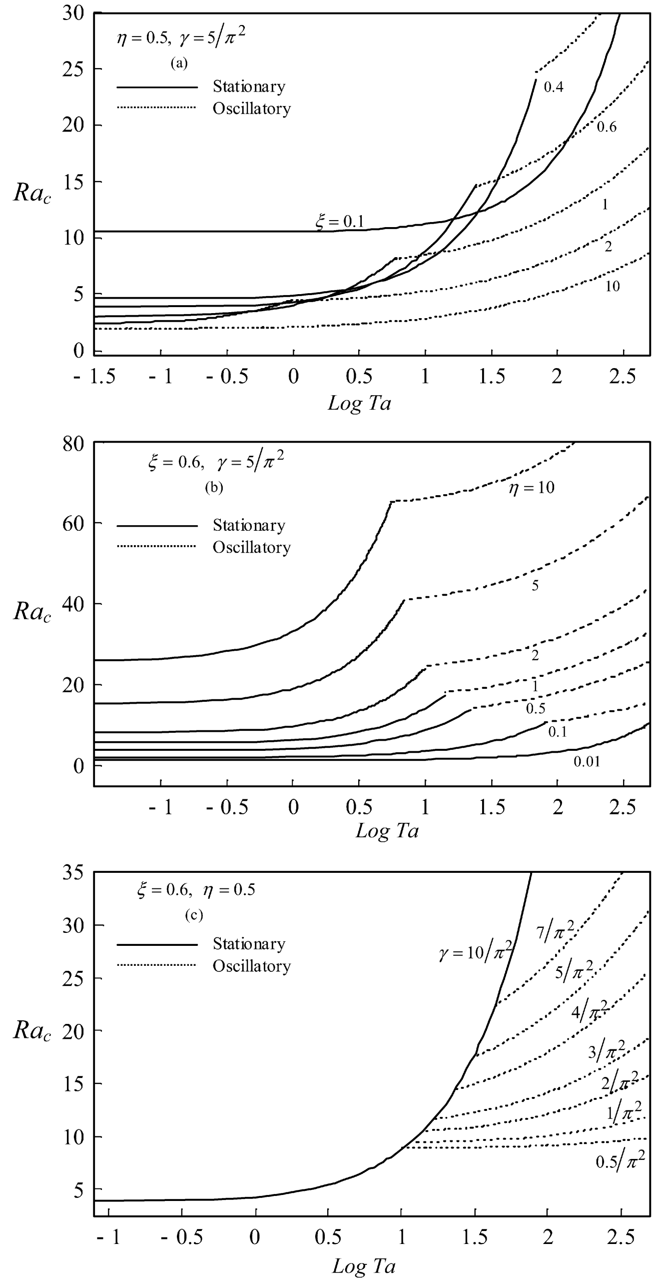

Fig. 6. Variation of scaled critical Rayleigh number with Taylor number for different values of (a) ξ, (b) η, and (c) γ.

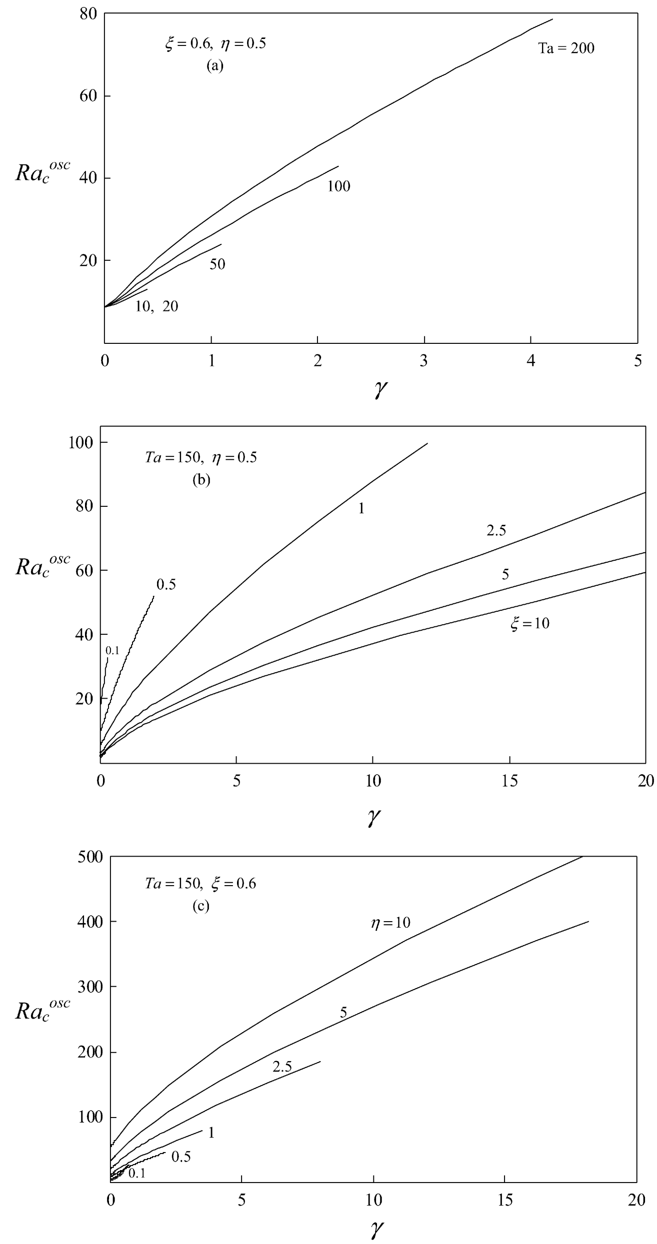

Fig. 7. Variation of scaled critical oscillatory Rayleigh number scaled Darcy–Prandtl number for different values of (a) Ta, (b) ξ and (c) η.

drashekhar [25]). However, this effect is not significant for the smaller values of the *Ta*. From each of these figures it is also observed that the convection first sets in as a stationary mode and after some value of $Ta > Ta^*$ (critical value) which depends on the other parameters such as ξ, η, and γ the instability manifests as oscillatory convection. Therefore, the oscillatory mode is the most dangerous mode for the system with moderate and higher values of *Ta*. Further it is interesting to note that for larger values of the mechanical anisotropy parameter ξ, the oscillatory convection exists even for small values of the Taylor number (Fig. 6(a)).

In Fig. 6(a) the variation of Ra_c with *Ta* for different values of mechanical anisotropy parameter ξ is shown for the fixed values $\eta = 0.5$ and $\gamma = 5/\pi^2$. It is important to note that the critical Rayleigh number for the direct mode decreases with increase of ξ for smaller values of *Ta* whereas for the higher values of the *Ta* this trend reverses. The critical Rayleigh number for the overstable mode always decreases with increase in the value of ξ. Further it is important to note that the value of the Taylor number *Ta*, at which the transition from stationary to oscillatory mode takes place decreases with the increase of ξ. Therefore increasing ξ increases the possibility of overstable motions even for small values of the Taylor number.

The effect of thermal anisotropy parameter η and scaled Darcy–Prandtl number γ on the onset criteria is shown in Figs. 6(b) and 6(c) respectively. We observe from these figures that effect of both η and γ is to delay the onset of convection

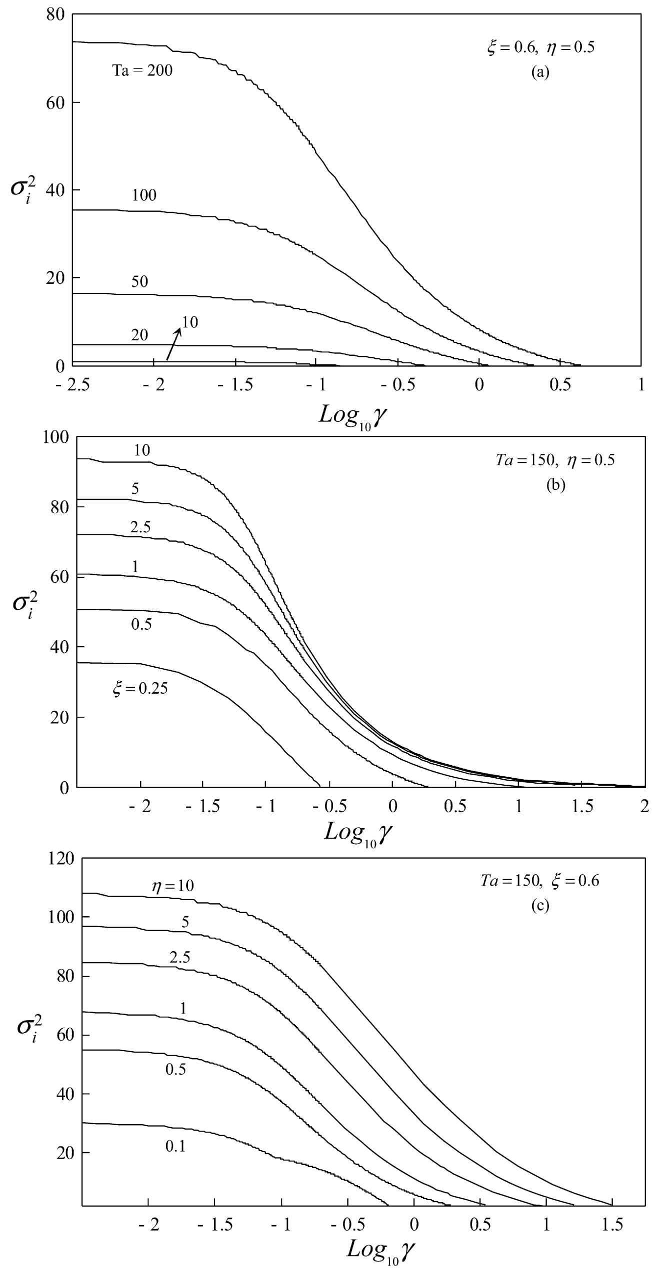

Fig. 8. Variation of critical frequency of oscillations with scaled Darcy–Prandtl number for different values of (a) *Ta*, (b) ξ and (c) η.

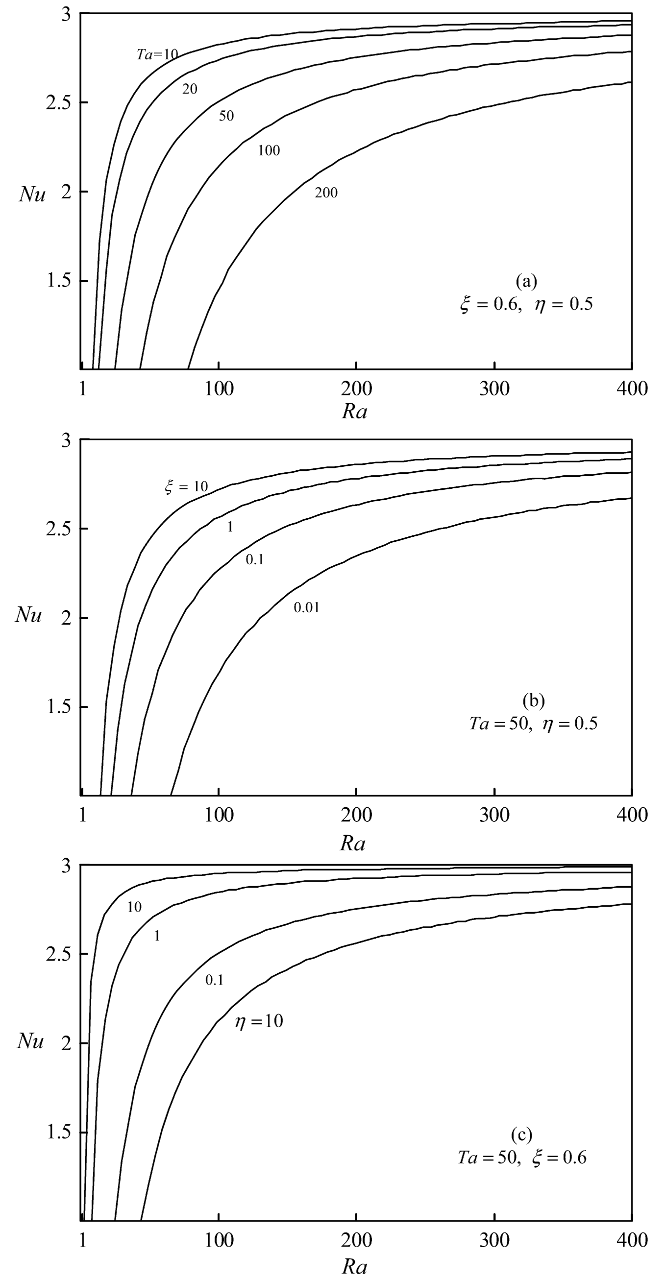

Fig. 9. Variation of Nusselt number *Nu* with scaled Rayleigh number *Ra* for different values of (a) *Ta*, (b) ξ and (c) η.

both in stationary and oscillatory modes. Further the value of *Ta* at which the transition from stationary to oscillatory mode occurs is found to decrease with η and increase with γ. Further Fig. 6(c) indicates that the oscillatory Rayleigh number is independent of the Taylor number when $\gamma \to 0$. This is consistent with the limiting cases discussed in Section 3.

The variation of oscillatory critical Rayleigh number Ra_c^{osc} as a function of γ for different values of Taylor number *Ta*, the anisotropy parameters ξ and η is shown in Figs. 7(a)–(c). The curves in each of these cases end at the point where no more values consistent with the condition $\sigma_i^2 > 0$ exist. These figures also indicate the range of values of Taylor number *Ta*, mechanical and thermal anisotropy parameter ξ and η for which the oscillatory motions exists. It is important to note that the range of γ over which the overstable motions are possible increases with Taylor number *Ta*, mechanical and thermal anisotropy parameters ξ and η.

The variation of the critical value of the frequency σ_{ic}^2 with γ for different values of Taylor number *Ta*, the anisotropy parameters ξ and η are shown respectively in Figs. 8(a)–(c). We observe that there is a marked increase in the value of the frequency with increasing Taylor number *Ta* and anisotropy parameters ξ and η when the values of γ are small and the frequencies decay as γ increases.

The quantity of heat transferred across the layer is computed by the Nusselt number as a function of *Ra*, *Ta*, ξ and η. The domain of nonlinear convection warrants the quantification of heat transfer. This is depicted in the Rayleigh–Nusselt number plane through Figs. 9(a)–(c). These figures reveal the quantitative effect of rotation and anisotropy of the porous layer on heat transport. We observe that as Ra increases from one to three times of its critical value, the heat transport increases sharply and as Ra is increased further, it remains almost constant. It is also found that the heat transport increases with increase in ξ where as it decreases with the increase in Ta and η. This is because the effect of rotation and thermal anisotropy is to inhibit the onset of thermal convection.

6. Conclusions

The onset of thermal convection in a fluid saturated rotating anisotropic porous layer is investigated using both linear and nonlinear stability analyses. The linear theory provides the criteria for the onset of stationary and oscillatory convection and the nonlinear theory, which is based on the truncated Fourier series method, provides a method to measure the convection amplitudes and the rate of heat transfer. The following conclusions are drawn:

(1) The oscillatory mode is most favorable for a system with moderate and high values of the Taylor number. However for large value of the mechanical anisotropy, the oscillatory motions exist even for small values of the Taylor number.
(2) The value of Taylor number at which the transition from stationary mode to the oscillatory mode takes place decreases with increase in the value of the mechanical anisotropy parameter.
(3) The effect of increasing the value of mechanical anisotropy parameter in the presence of rotation is to allow the onset of convection to be oscillatory rather than stationary.
(4) The effect of Darcy–Prandtl number is to delay the onset of oscillatory convection.
(5) The Nusselt number increases with increase in ξ where as it decreases with the increase in Ta and η.

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